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DECAY OF WAVES IN A ROD OWING TO THE HYSTERESIS-TYPE LOSS OF ENERGY IN A MATERIAL

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Decay of waves propagating in a semi-infinite rod owing to the hysteresis-type loss of strain energy in the material is studied. Results of numerical calculations are given.

Key words: *rod, waves, hysteresis, decay.*

Wave processes in a semi-infinite rod located in an elastic medium under the action of a pulsed load $p(x,t)$ linearly distributed at the initial segment $0 \leqslant x \leqslant a$ were considered in [1]. Let us write the equations of motion with allowance for the shear strain and rotation inertia:

$$\frac{\partial Q}{\partial x} + \alpha W = p(x,t) - \rho F \frac{\partial^2 W}{\partial t^2}, \qquad \frac{\partial M}{\partial x} - Q = \rho I \frac{\partial^2 \theta}{\partial t^2}. \tag{1}$$

Here, $Q = k'GF(\theta - \partial W/\partial x)$ and $M = EJ\,\partial\theta/\partial x$ are the transverse force and the bending moment, θ and W are the angle of rotation and deflection, and x and t are the longitudinal coordinate and time, respectively. In this paper, we consider decay of waves in the rod owing to the hysteresis-type loss of energy in the material.

The stress–strain dependences that describe the hysteresis loop can be presented in the following form [2]:

$$\overset{\rightleftarrows}{\sigma} = E\Big[\chi \pm \frac{3\delta_1}{8}\Big(\chi_a \mp 2\chi - \frac{\chi^2}{\chi_a}\Big)\Big]z, \qquad \overset{\rightleftarrows}{\tau} = k'G\Big[\varkappa \pm \frac{3\delta_2}{8}\Big(\varkappa_a \mp 2\varkappa - \frac{\varkappa^2}{\varkappa_a}\Big)\Big]. \tag{2}$$

Here, χ and $\varkappa$ are the tension–compression and shear strains along the neutral line, χ_a and $\varkappa_a$ are their amplitudes, and δ_1 and δ_2 are the decrements of the decay of oscillations due to bending and shear, respectively. It writing Eqs. (2), we assumed that the energy scattering does not depend on the normal and shear stresses.

Using Eqs. (1), we find the tension–compression and shear strains:

$$\chi = \Big(\frac{\partial^2 W}{\partial x^2} - \frac{\rho}{k'G}\frac{\partial^2 W}{\partial t^2}\Big)z, \qquad \varkappa = \frac{J}{k'GF}\Big(\rho\frac{\partial^2\theta}{\partial t^2} - E\frac{\partial^2\theta}{\partial x^2}\Big).$$

Using Eqs. (2), we find the moment and transverse force in the rod made of a material with imperfect elasticity:

$$M = EJ\frac{\partial\theta}{\partial x} + \varepsilon\,\overset{\rightleftarrows}{\Phi}, \qquad Q = k'GF\Big(\theta - \frac{\partial W}{\partial x}\Big) + \varepsilon\,\overset{\rightleftarrows}{\Psi}. \tag{3}$$

Here,

$$\varepsilon\,\overset{\rightleftarrows}{\Phi} = \pm\frac{3\delta_1}{8}E\int\limits_F\Big(\chi_a \mp 2\chi - \frac{\chi^2}{\chi_a}\Big)z\,dF, \qquad \varepsilon\,\overset{\rightleftarrows}{\Psi} = \pm\frac{3\delta_2}{8}k'G\int\limits_F\Big(\varkappa_a \mp 2\varkappa - \frac{\varkappa^2}{\varkappa_a}\Big)dF$$

are the functionals that take into account energy dissipation due to internal friction in the material and ε is a small parameter that ensures small values of the functionals and weak nonlinearity of the differential equations. The right-turned arrow above σ, τ, Φ, and Ψ and the upper signs in the right sides of Eqs. (2) and (3) refer to the uprising branch of the hysteresis loop, while the left-turned arrow and the lower signs refer to the downward branch.

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Substituting Eq. (3) into Eq. (1), we obtain the equations of rod motion, which take into account hysteresis-type scattering of energy and contain the decrement of decay of oscillations.

Using the dimensionless quantities

$$\xi = \frac{x}{r}, \quad w = \frac{W}{r}, \quad \tau = \frac{c_1 t}{r}, \quad \beta = \frac{r}{a}, \quad r^2 = \frac{J}{F}, \quad m = \frac{Mr}{EJ},$$

we write the dimensionless equations of motion in displacements as

$$\begin{gathered}\frac{\partial^2 w}{\partial \xi^2} - \frac{\partial \theta}{\partial \xi} - \zeta w - \gamma \frac{\partial^2 w}{\partial \tau^2} - \varepsilon \frac{\partial \Psi^*}{\partial \xi} = -k(1 - \beta\xi)H(\tau),\\ \frac{\partial w}{\partial \xi} - \theta - \gamma\Big(\frac{\partial^2 \theta}{\partial \xi^2} - \frac{\partial^2 \theta}{\partial \tau^2}\Big) + \varepsilon\gamma \frac{\partial \Phi^*}{\partial \xi} - \varepsilon \Psi^* = 0,\end{gathered} \tag{4}$$

where

$$\begin{gathered}\varepsilon\Phi^* = \pm\frac{3\delta_1}{8}\Big(\chi_a^* \mp 2\chi^* - \frac{\chi^{*2}}{\chi_a^*}\Big), \qquad \varepsilon\Psi^* = \pm\frac{3\delta_2}{8}\Big(\varkappa_a^* \mp 2\varkappa^* - \frac{\varkappa^{*2}}{\varkappa_a^*}\Big),\\ \chi^* = \frac{\partial^2 w}{\partial \xi^2} - \gamma \frac{\partial^2 w}{\partial \tau^2}, \qquad \varkappa^* = \gamma\Big(\frac{\partial^2 \theta}{\partial \tau^2} - \frac{\partial^2 \theta}{\partial \xi^2}\Big),\end{gathered} \tag{5}$$

$\gamma = c_1^2/c_2^2$, $\zeta = \alpha r^2/(\rho F c_2^2)$, $k = p_0 r/(\rho F c_2^2)$, and $c_1^2 = E/\rho$ and $c_2^2 = k'G/\rho$ are the velocities of propagation of the streamwise and transverse waves, respectively. In Eqs. (4) and below, the arrows above Φ^* and Ψ^* are omitted.

We solve Eqs. (4) using the method of expansion into a series in powers of the small parameter [2, 3]. Confining ourselves to the first approximation, we write

$$w(\xi,\tau) = u\cos\varphi + \varepsilon w_1(u,\varphi) + \varepsilon^2 \ldots\ , \qquad \theta(\xi,\tau) = v\cos\psi + \varepsilon\theta_1(v,\psi) + \varepsilon^2 \ldots\ . \tag{6}$$

Here, $w_1(u,\varphi), w_2(u,\varphi), \ldots$ and $\theta_1(v,\psi), \theta_2(v,\psi), \ldots$ are periodic functions with a period 2π.

The functionals Φ^* and Ψ^* are also written in the form of an expansion into a Taylor series [2]:

$$\varepsilon\Phi^*(w) = \varepsilon\Phi^*(u,\cos\varphi) + \varepsilon^2\Phi_u^{*\prime}(u,\cos\varphi)w_1 \ldots + \varepsilon^3 \ldots\ ,$$

$$\varepsilon\Psi^*(\theta) = \varepsilon\Psi^*(v,\cos\psi) + \varepsilon^2\Psi_v^{*\prime}(v,\cos\psi)\theta_1 \ldots + \varepsilon^3 \ldots\ .$$

The amplitudes u and v and the phases φ and ψ involved into series (6) are functions of time and are determined from the differential equations

$$\begin{gathered}\frac{du}{d\tau} = \varepsilon A_1(u) + \varepsilon^2 A_3(u) + \ldots, \qquad \frac{d\varphi}{d\tau} = a_3 + \varepsilon B_1(u) + \varepsilon^2 \ldots\ ,\\ \frac{dv}{d\tau} = \varepsilon A_2(v) + \varepsilon^2 A_4(v) + \ldots, \qquad \frac{d\psi}{d\tau} = \beta_1 + \varepsilon B_2(u) + \varepsilon^2 \ldots\ .\end{gathered} \tag{7}$$

It follows from Eqs. (7) that the amplitudes decrease owing to energy scattering, and the instantaneous frequencies depend on the amplitudes.

The first terms in the right sides of Eqs. (6) are solutions of the zeroth approximation equations (see [1]). In the variables used in [1], the expressions for the deflection and bending moment have the form

$$w^0(\xi,\tau) = u\cos\varphi, \qquad m(\xi,\tau) = \frac{\partial\theta^0}{\partial\xi} = -\frac{k}{\gamma\zeta\nu}\sin\beta_1\xi, \tag{8}$$

where $u = k\gamma(1-\zeta)/(2\zeta^2)$, $\varphi = a_3\xi$, $a_3 = (\zeta/\gamma)^{1/2}$, $\beta_1 = [(a_3 - \zeta/2)/2]^{1/2}$, and $\nu = [4/(\gamma\zeta) - 1]^{1/2}$.

Integrating the second expression in (8), we find

$$\theta^0(\xi,\tau) = v\cos\psi, \qquad \psi = \beta_1\xi, \tag{9}$$

where $v = 2k/(\gamma\zeta\nu\beta_1)$ at $0 \leqslant \xi \leqslant \xi_2$ and $v = k/(\gamma\zeta\nu\beta_1)$ at $\xi_2 \leqslant \xi \leqslant \xi_1$.

The coordinate ξ and the time τ are related as

$$\xi = \begin{cases} \tau/\sqrt{\gamma}, & 0 \leqslant \xi \leqslant \xi_2,\\ \tau, & \xi_2 \leqslant \xi \leqslant \xi_1.\end{cases}$$

Using Eqs. (5), (8), and (9), we find the strains and the functionals

$$\chi^* = (\gamma - 1)a_3^2 u \cos\varphi, \qquad \varkappa^* = (\gamma - 1)\beta_1^2 v \cos\psi,$$

$$\varepsilon\Phi^* = \pm\frac{3}{8}\,\delta_1(\gamma - 1)a_3^2 u(1 \mp 2\cos\varphi - \cos^2\varphi), \quad \varepsilon\Psi^* = \pm\frac{3}{8}\,\delta_2(\gamma - 1)\beta_1^2 v(1 \mp 2\cos\psi - \cos^2\psi).$$

From Eqs. (4), we find two fourth-order differential equations with respect to w and θ. Differentiating the right side of Eqs. (6) with respect to time, taking into account Eqs. (7), substituting the resultant expressions into the fourth-order equations with respect to w and θ, and retaining terms containing the small parameter with powers not greater than one, we obtain

$$\frac{\partial^4 w_1}{\partial\xi^4} - \zeta\,\frac{\partial^2 w_1}{\partial\xi^2} + \frac{\zeta}{\gamma}\,w_1 - (\gamma + 1)\,\frac{\partial^4 w_1}{\partial\xi^2\,\partial\tau^2} + (1 + \zeta)\,\frac{\partial^2 w_1}{\partial\tau^2} + \gamma\,\frac{\partial^4 w_1}{\partial\tau^4}$$

$$= 2a_3(1 - \zeta)(A_1\sin\varphi + uB_1\cos\varphi) - \gamma\Big(\frac{\partial^3\Psi^*}{\partial\xi\partial\tau^2} - \frac{\partial^3\Psi^*}{\partial\xi^3}\Big) - \frac{\partial^2\Phi^*}{\partial\xi^2},$$

$$\frac{\partial^4\theta_1}{\partial\xi^4} - \zeta\,\frac{\partial^2\theta_1}{\partial\xi^2} - \frac{\zeta}{\gamma}\,\theta_1 - (\gamma + 1)\,\frac{\partial^4\theta_1}{\partial\xi^2\,\partial\tau^2} - (1 - \zeta)\,\frac{\partial^2\theta_1}{\partial\tau^2} + \gamma\,\frac{\partial^4\theta_1}{\partial\tau^4} \tag{10}$$

$$+ 2\beta_1(A_2\sin\psi + vB_2\cos\psi) = \frac{\zeta}{\gamma}\,\Psi^* + \frac{\partial^2\Psi^*}{\partial\tau^2} - \zeta\,\frac{\partial\Phi^*}{\partial\xi} - \gamma\,\frac{\partial^3\Phi^*}{\partial\xi\,\partial\tau^2} + \frac{\partial^3\Phi^*}{\partial\xi^3}.$$

Equations of the second approximation are not given in the present paper.

According to the theory of constructing asymptotic solutions [3], we impose restrictions on the functions $w_1(u,\varphi), \theta_1(v,\psi), \ldots, A_1(u), B_1(u), \ldots$ and $A_2(v), B_2(v), \ldots$ in the form

$$\int_0^{2\pi} w_i(u,\varphi)\left\{\begin{array}{c}\cos\varphi\\ \sin\varphi\end{array}\right\}d\varphi = 0, \qquad \int_0^{2\pi} \theta_i(v,\psi)\left\{\begin{array}{c}\cos\psi\\ \sin\psi\end{array}\right\}d\psi = 0.$$

This means that the amplitudes u and v are taken in expansions (6) as the total amplitudes of the fundamental harmonics of oscillations. Let us multiply the first equation in (10) by $\cos\varphi\,d\varphi$ and $\sin\varphi\,d\varphi$, multiply the second equation by $\cos\psi\,d\psi$ and $\sin\psi\,d\psi$, and perform integration. In calculating the integrals, we use the formulas of integration by parts and the rule of differentiation under the integral sign. As a result of these calculations, we obtain

$$\varepsilon A_1 = -\frac{3(\gamma - 1)a_3^3\delta_1 u}{4\pi(1 - \zeta)}, \qquad \varepsilon B_1 = \frac{3(\gamma - 1)a_3^3\delta_1 u}{8(1 - \zeta)},$$

$$\varepsilon A_2 = -\frac{3\delta_2(\gamma - 1)\beta_1 v}{16\pi}\Big(\frac{3\zeta}{4\gamma} + 4\beta_1^2\Big), \qquad \varepsilon B_2 = \frac{3\delta_2(\gamma - 1)\beta_1}{8}\Big(-\frac{\zeta}{\gamma} + \beta_1^2\Big). \tag{11}$$

Substituting Eqs. (11) into Eqs. (7), integrating, and requiring the initial conditions to be satisfied, we obtain

$$u = u_0\,\mathrm{e}^{-\eta_1\delta_1\tau}\cos[(1 + \mu_1\delta_1)a_3\tau], \qquad v = v_0\,\mathrm{e}^{-\eta_2\delta_2\tau}\cos[(1 + \mu_2\delta_2)\beta_1\tau], \tag{12}$$

where $\eta_1 = 3(\gamma - 1)a_3^3/[4\pi(1 - \zeta)]$, $\mu_1 = 3(\gamma - 1)\zeta/(8\gamma)$, $\eta_2 = 3(\gamma - 1)\beta_1^3/(4\pi)$, $\mu_2 = 3(\gamma - 1)\beta_1^2/8$, $u_0 = u|_{\tau=0}$, and $v_0 = v|_{\tau=0}$.

The bending moment is found by differentiating the expression for v with respect to ξ:

$$m = m_0\,\mathrm{e}^{-\eta_2\delta_2\tau}\sin[(1 + \mu_2\delta_2)\beta_1\tau], \qquad m_0 = v_0\beta_1. \tag{13}$$

The results of studying the convergence of series (6) shows that the terms εw_1 and $\varepsilon\theta_1$ are quantities of the first order of smallness; therefore, the solutions of the first approximation can be taken in the form $w = u\cos\varphi$ and $\theta = v\cos\psi$. In the explicit form, these solutions are similar to Eqs. (12) and (13). The accuracy of these engineering calculations is fairly sufficient [3]. The small parameter is not used in the calculations; therefore, its physical meaning is not clarified.

The calculations were performed for the following data: rectangular cross section of the rod with $b = h = 0.1$ m, $F = b \times h = 0.1 \times 0.1$ m, $p_0 = 40$ kN/m, rod material is the St. 45 steel, $E = 2 \cdot 10^5$ MPa,

$\rho = 8$ tons/m^3, $\zeta = 1.35 \cdot 10^{-2}$, $\varkappa = 1.8 \cdot 10^{-6}$, $\beta = 0.1$, and $\gamma = 3.1$. The stresses $\sigma = 58$ MPa and $\tau = 22$ MPa arising in the rod correspond to the logarithmic decrements of oscillations $\delta_1 = 0.3\%$ and $\delta_2 = 0.35\%$ [4]. With increasing stresses, the decrements of oscillations increase. As the external load is tripled, the stresses also increase by a factor of three, and the decrements of oscillations are $\delta_1 = 0.8\%$ and $\delta_2 = 0.7\%$. The amplitudes of the deflection and bending moment were determined by Eqs. (12) and (13).

The amplitudes of the waves of deflections and stresses decrease by a factor of 10 during the times $\tau = 4.9 \cdot 10^6$ and $\tau = 2.3 \cdot 10^5$, respectively. During these times, the fronts of the longitudinal waves cover distances equal to 140 and 6.7 km, respectively. For the stresses increased by a factor of three, these distances are 50 and 4.8 km, respectively. At a distance of 10 km from the point of application of the external load, the deflection amplitude is 85% of the initial value, and the bending moment is 3.2%. The calculated values are $\mu_1 \delta_1 = 1.03 \cdot 10^{-5}$ and $\mu_2 \delta_2 = 0.91 \cdot 10^{-4}$, i.e., the frequencies of oscillations change insignificantly.

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